A quick guide to sketching phase planes

Section 6.1 of the text discusses equilibrium points and analysis of the phase plane. However, there is one idea, not mentioned in the book, that is very useful to sketching and analyzing phase planes, namely *nullclines*. Recall the basic setup for an autonomous system of two DEs:

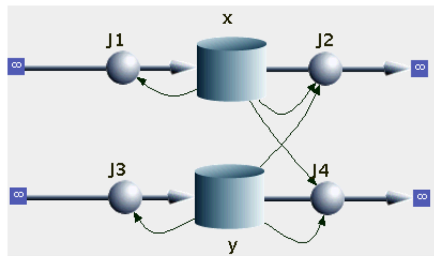
$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

To sketch the phase plane of such a system, at each point (x_0, y_0) in the xy-plane, we draw a vector starting at (x_0, y_0) in the direction $f(x_0, y_0)\mathbf{i} + g(x_0, y_0)\mathbf{j}$.

Definition of nullcline. The x**-nullcline** is a set of points in the phase plane so that $\frac{dx}{dt} = 0$. Geometrically, these are the points where the vectors are either straight up or straight down. Algebraically, we find the x-nullcline by solving $f(x,y) = 0$.

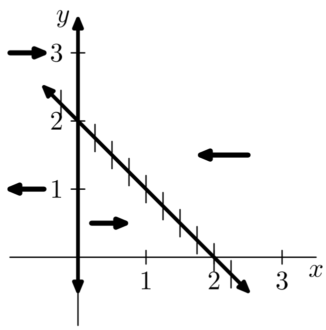
The y**-nullcline** is a set of points in the phase plane so that $\frac{dy}{dt} = 0$. Geometrically, these are the points where the vectors are horizontal, going either to the left or to the right. Algebraically, we find the y-nullcline by solving $g(x,y) = 0$.

How to use nullclines. Consider the system

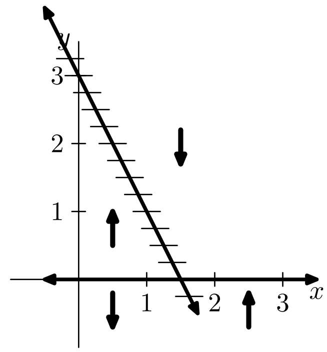


$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy,$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

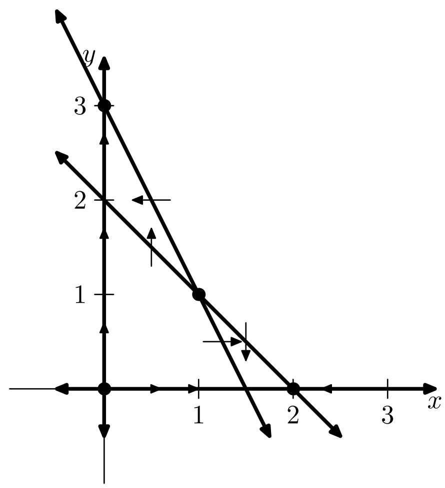
To find the x-nullcline, we solve $2x\left(1 - \frac{x}{2}\right) - xy = 0$, where multiplying out and collecting the common factor of x gives $x(2 - x - y) = 0$. This gives two x-nullclines, the line $x + y = 2$ and the y-axis. By plugging in the points (1,0) and (2,2) into $2x\left(1 - \frac{x}{2}\right) - xy$, we see that solutions (to the left of the y-axis) move to the right if below the line $x + y = 2$ and to the left if above it.



To find the y-nullcline, we solve $3y\left(1 - \frac{y}{3}\right) - 2xy = 0$, where multiplying out and collecting the common factor of y gives $y(3 - y - 2x) = 0$. This gives two y-nullclines, the line $2x + y = 3$ and the x-axis. By plugging in the points (0,1) and (2,2) into $3y\left(1 - \frac{y}{2}\right) - 2xy$, we see that solutions (above the x-axis) move up if below the line $2x + y = 3$ and down if above it.



Combining this information gives us the following picture. Notice that we can draw directions on each nullcline by using the direction information from the other graph. For example, the line segment from $(1,1)$ to $(0,3)$, since it is above the line $x+y=2$, has solutions moving to the left.



Also, where the x-nullcline and y-nullcline cross, both $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are zero. So these points (marked by dots in the above graph) are equilibrium points.

Once a solution enters the triangle with vertices $(1,1)$, $(0,2)$ and $(0,3)$, it can never leave. Similarly, solutions in the triangle with vertices $(1,1)$, $(3/2,0)$ and $(2,0)$ can never leave.

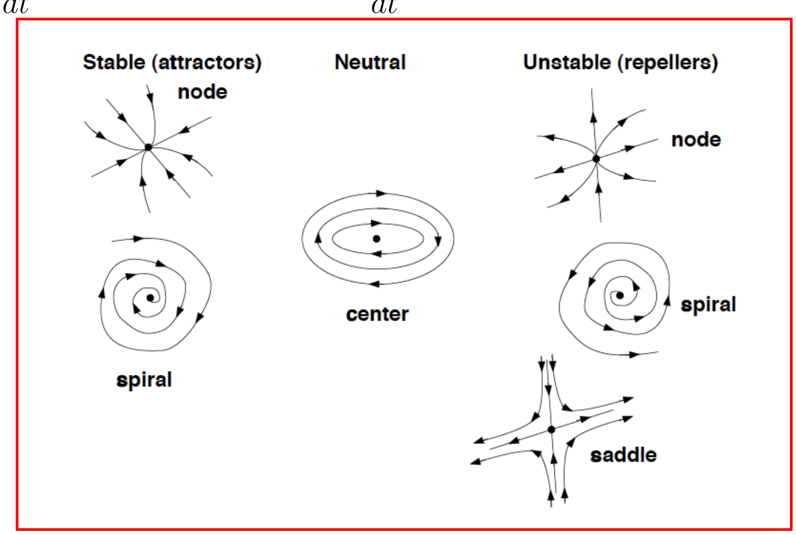
Exercises. Graph the nullclines and discuss the possible fates of solutions for the following systems. The nullclines may not be straight lines.

(1) $\dfrac{dx}{dt} = x(-x-3y+150), \quad \dfrac{dy}{dt} = y(-2x-y+100).$

(2) $\dfrac{dx}{dt} = x(10-x-y), \quad \dfrac{dy}{dt} = y(30-2x-y).$

(3) $\dfrac{dx}{dt} = 2x\left(1-\dfrac{x}{2}\right) - xy, \quad \dfrac{dy}{dt} = y\left(\dfrac{9}{4}-y^2\right) - x^2y.$

(4) $\dfrac{dx}{dt} = x(-4x-y+160), \quad \dfrac{dy}{dt} = y(-x^2-y^2+2500).$



Qualitative Analysis

Very often it is almost impossible to find explicitly of implicitly the solutions of a system (specially nonlinear ones). The qualitative approach as well as numerical one are important since they allow us to make conclusions regardless whether we know or not the solutions.

Recall what we did for autonomous equations. First we looked for the equilibrium points and then, in conjunction with the existence and uniqueness theorem, we concluded that non-equilibrium solutions are either increasing or decreasing. This is the result of looking at the sign of the derivative. So what happened for autonomous systems? First recall that the components of the velocity vectors are $\frac{dx}{dt}$ and $\frac{dy}{dt}$. These vectors give the direction of the motion along the trajectories. We have the four natural directions (left-down, left-up, right-down, and right-up) and the other four directions (left, right, up, and down). These directions are obtained by looking at the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and whether they are equal to 0. If both are zero, then we have an equilibrium point.

Example. Consider the model describing two species competing for the same prey

$$\left\{\begin{array}{lcl} \dfrac{dx}{dt} & = & x(1-x) - xy \\ \\ \dfrac{dy}{dt} & = & 2y(1-y/2) - 3xy \end{array}\right.$$

Let us only focus on the first quadrant $x \geq 0$ and $y \geq 0$. First, we look for the equilibrium points. We must have

$$\left\{\begin{array}{lcl} x(1-x) - xy & = & 0 \\ 2y(1-y/2) - 3xy & = & 0 \end{array}\right.$$

Algebraic manipulations imply

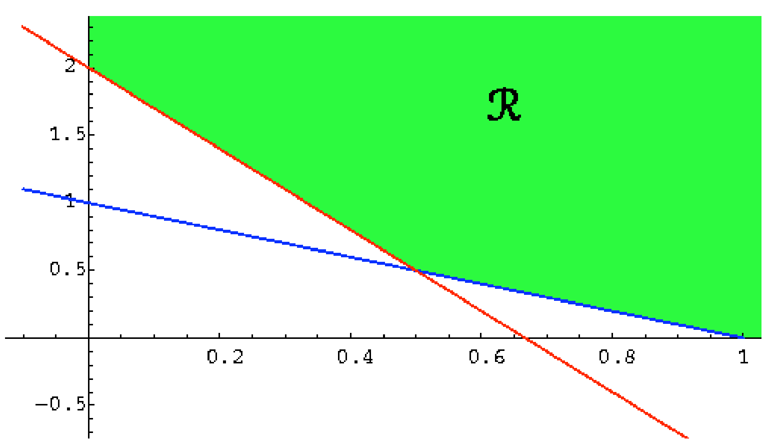
$$x = 0 \ \text{ or } \ 1 - x - y = 0$$

and

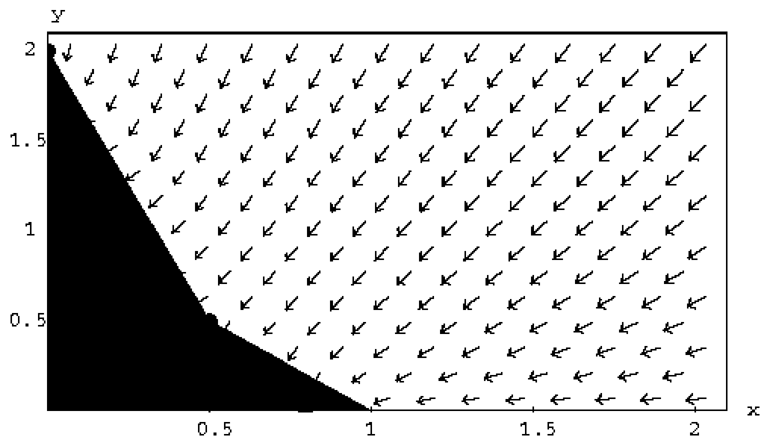
$$y = 0 \ \text{ or } \ 2 - 3x - y = 0$$

The equilibrium points are (0,0), (0,2), (1,0), and $\left(\frac{1}{2}, \frac{1}{2}\right)$.

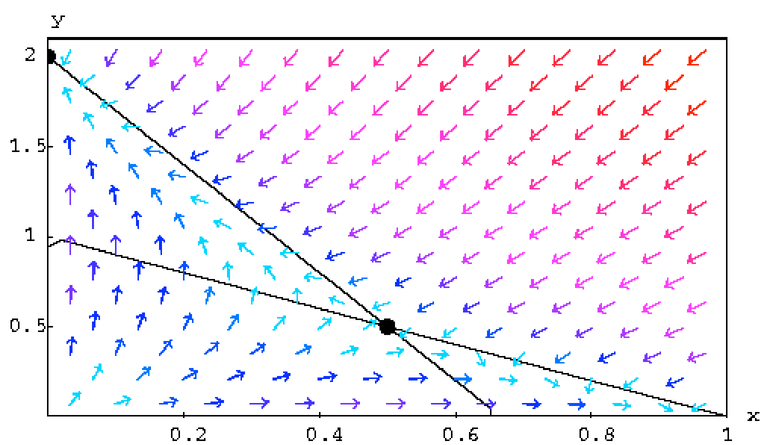
Consider the region **R** delimited by the x-axis, the y-axis, the line 1-*x*-*y*=0, and the line 2-3*x*-*y*=0.



Clearly inside this region neither $\frac{dx}{dt}$ or $\frac{dy}{dt}$ are equal to 0. Therefore, they must have constant sign (they are both negative). Hence the direction of the motion is the same (that is left-down) as long as the trajectory lives inside this region.



In fact, looking at the first-quadrant, we have three more regions to add to the above one. The direction of the motion depends on what region we are in (see the picture below)



The boundaries of these regions are very important in determining the direction of the motion along the trajectories. In fact, it helps to visualize the trajectories as slope-field did for autonomous equations. These boundaries are called **nullclines**.

Nullclines.

Consider the autonomous system

$$\begin{cases} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{cases}$$

The **x-nullcline** is the set of points where $\frac{dx}{dt} = 0$ and **y-nullcline** is the set of points where $\frac{dy}{dt} = 0$. Clearly the points of intersection between x-nullcline and y-nullcline are exactly the equilibrium points. Note that along the x-nullcline the velocity vectors are vertical while along the y-nullcline the velocity vectors are horizontal. Note that as long as we are traveling along a nullcline without crossing an equilibrium point, then the direction of the velocity vector must be the same. Once we cross an equilibrium point, then we may have a change in the direction (from up to down, or right to left, and vice-versa).

Example. Draw the nullclines for the autonomous system and the velocity vectors along them.

$$\left\{\begin{array}{rcl}\dfrac{dx}{dt} & = & x(1-x) - xy\\[2ex]\dfrac{dy}{dt} & = & 2y(1-y/2) - 3xy\end{array}\right.$$

The x-nullcline are given by

$$\frac{dx}{dt} = x(1-x) - xy = 0$$

which is equivalent to

$$x = 0 \text{ or } 1 - x - y = 0$$

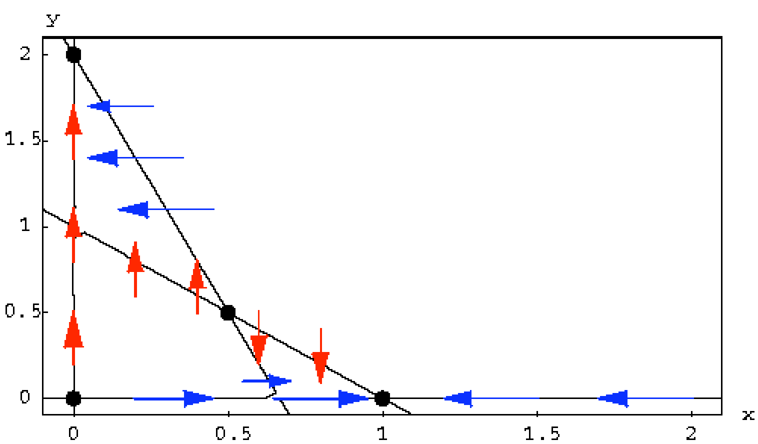
while the y-nullcline are given by

$$\frac{dy}{dt} = 2y(1-y/2) - 3xy = 0$$

which is equivalent to

$$y = 0 \text{ or } 2 - 3x - y = 0$$

In order to find the direction of the velocity vectors along the nullclines, we pick a point on the nullcline and find the direction of the velocity vector at that point. The velocity vector along the segment of the nullcline delimited by equilibrium points which contains the given point will have the same direction. For example, consider the point (2,0). The velocity vector at this point is (-1,0). Therefore the velocity vector at any point (x,0), with $x > 1$, is horizontal (we are on the y-nullcline) and points to the left. The picture below gives the nullclines and the velocity vectors along them.



In this example, the nullclines are lines. In general we may have any kind of curves.

Example. Draw the nullclines for the autonomous system

$$\left\{\begin{array}{rcl}\dfrac{dx}{dt} & = & x(1-x) - xy\\[2ex]\dfrac{dy}{dt} & = & 2y(1-y^2/2) - 3x^2y\end{array}\right.$$

The x-nullcline are given by

$$\frac{dx}{dt} = x(1-x) - xy = 0$$

which is equivalent to

$$x = 0 \text{ or } 1 - x - y = 0$$

while the y-nullcline are given by

$$\frac{dy}{dt} = 2y(1-y^2/2) - 3x^2y = 0$$

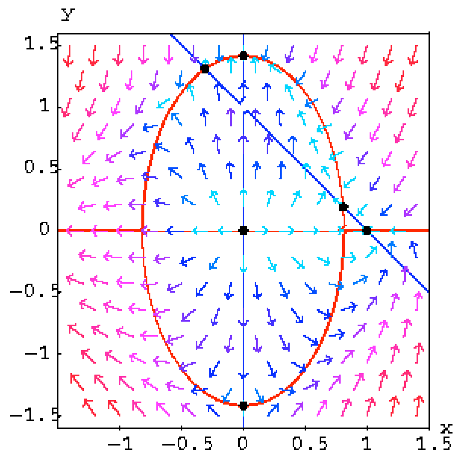
$$\frac{dy}{dt} = 2y(1 - y^2/2) - 3x^2y = 0$$

which is equivalent to

$$y = 0 \;\; \text{or} \;\; 2 - 3x^2 - y^2 = 0$$

Hence the y-nullcline is the union of a line with the ellipse

$$3x^2 + y^2 = 2$$



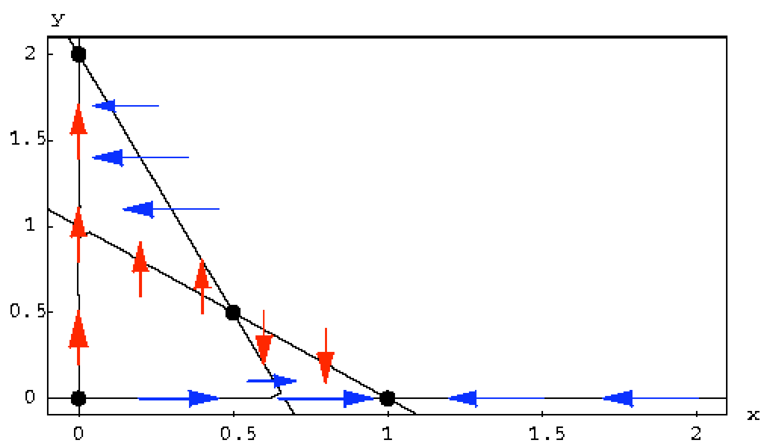
Information from the nullclines

For most of the nonlinear autonomous systems, it is impossible to find explicitly the solutions. We may use numerical techniques to have an idea about the solutions, but qualitative analysis may be able to answer some questions with a low cost and faster than the numerical technique will do. For example, questions related to the long term behavior of solutions. The nullclines plays a central role in the qualitative approach. Let us illustrate this on the following example.

Example. Discuss the behavior of the solutions of the autonomous system

$$\begin{cases} \dfrac{dx}{dt} & = & x(1-x) - xy \\ \\ \dfrac{dy}{dt} & = & 2y(1-y/2) - 3xy \end{cases}$$

We have already found the nullclines and the direction of the velocity vectors along these nullclines.

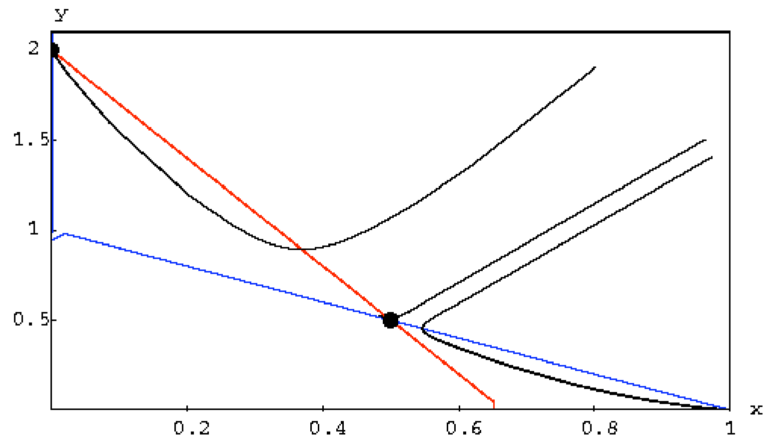


These nullclines give the birth to four regions in which the direction of the motion is constant. Let us discuss the region bordered by the x-axis, the y-axis, the line 1-*x*-*y*=0, and the line 2-3*x*-*y*=0. Then the direction of the motion is left-down. So a moving object starting at a position in this region, will follow a path going left-down. We have three choices

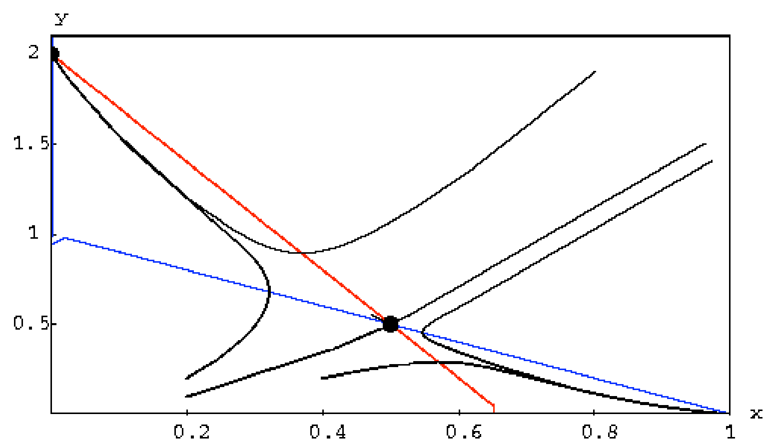
- **First choice:** the trajectory dies at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

- **Second choice:** the starting point is above the trajectory which dies at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Then the trajectory will hit the triangle defined by the points $\left(\frac{1}{2}, \frac{1}{2}\right)$, (0,1), and (0,2). Then it will go up-left and dies at the equilibrium point (0,2).

- **Third choice:** the starting point is below the trajectory which dies at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Then the trajectory will hit the triangle defined by the points $\left(\frac{1}{2}, \frac{1}{2}\right)$, (1,0), and $\left(\frac{2}{3}, 0\right)$. Then it will go down-right and dies at the equilibrium point (1,0).



For the other regions, look at the picture below. We included some solutions for every region.



Remarks. We see from this example that the trajectories which dye at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$ are crucial to predicting the behavior of the solutions. These two trajectories are called **separatrix** because they separate the regions into different subregions with a specific behavior. To find them is a very difficult problem. Notice also that the equilibrium points (0,2) and (1,0) behave like sinks. The classification of equilibrium points will be discussed using the approximation by linear systems.

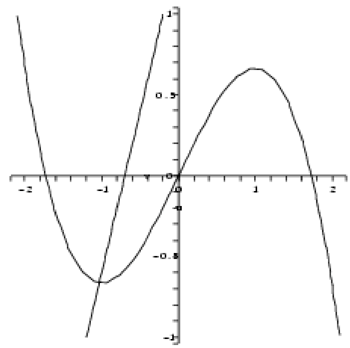
The FitzHugh-Nagumo model which we will use for the analysis is:

$$\begin{aligned} \dot{u} &= \frac{(v + u - u^3/3)}{\epsilon} \\ \dot{v} &= \epsilon(u + \beta - \gamma v) \end{aligned}$$

The following formulas

$$\begin{aligned} v &= \frac{1}{3}u^3 - u \\ v &= \frac{1}{\gamma}u + \frac{\beta}{\gamma} \end{aligned}$$

are the u and v nullcline respectively. *(See graph of the nullcline below.)*



Graph of the nullcline with parameter *($\epsilon = 0.2, \beta = 0.7, \gamma = 0.5$)*

The nullclines $v = \frac{1}{3}u^3 - u$ and $v = \frac{1}{\gamma}u + \frac{\beta}{\gamma}$ represent the set of points when the change with repect to time of u and v is null. Hence the solution will cross the 'cubic' nullcline *(i.e. u nullcline)* when travelling vertically while the solution will cross the 'line' nullcline *(i.e. v nullcline)* when travelling horizontally. Suppose that we fix the parameters of the FitzHugh-Nagumo model to $\epsilon = 0.2, \beta = 0.7, \gamma = 0.5$, then suppose that you let a solution start at $x_0 = (1, -0.5)$. Then we notice that $\frac{du}{dt} > 0$ and that $\frac{dv}{dt} > 0$, so the solution will travel upwards and to the right. It will only cross the 'cubic' nullcline vertically. Therefore once the solution will be close to the nullcline it will travel vertically and cross the nullcline. Once the solution has crossed the nullcline it will move upwards and to the left since $\frac{du}{dt} < 0$ and that $\frac{dv}{dt} > 0$. It will then cross the 'line' nullcline horizontally and the solution will move downwards and will remain moving towards the left since $\frac{du}{dt} < 0$ and $\frac{dv}{dt} < 0$, until the solution reaches the 'cubic' nullcline which it will cross vertically. Once the solution has crossed the last nullcline it will continue to move downwards but will now progressively start moving to the right since $\frac{du}{dt} > 0$ and $\frac{dv}{dt} < 0$. This is all that can be said about this specific case using the analysis of the nullcline. Below is an example of the solution. As we can see it intially behaves like the above description.