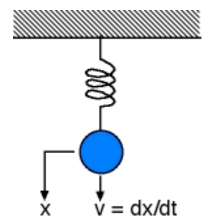
EXCITABLE & OSCILLATORY SYSTEMS

Excitability refers to the phenomenon where a system has but a single stable attractor, but it has two modes of returning to the equilibrium state. For small perturbations away from the equilibrium, the return is monotonic; however, for perturbations beyond a threshold value, the return is not monotonic, but undergoes a large excursion before settling down. The toilet is an example of an excitable system.

The harmonic oscillator

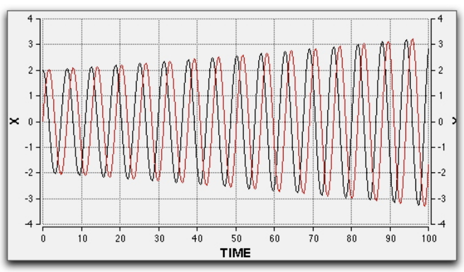
The simplest oscillator is a mass on a spring. The equation of motion is given by Newton's Law: F = m·a ⇒ F = m·d^2x/dt^2. Let m = 1, and v = dx/dt. The spring force is F = –k·x. Then the equations of motion become

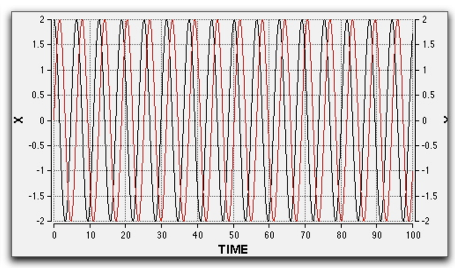


$$\frac{dx}{dt} = v,\ m\frac{dv}{dt} = -kx, \quad or \qquad \tau\frac{dv}{dt} = -x, \text{ where } \tau = \text{m/k}$$

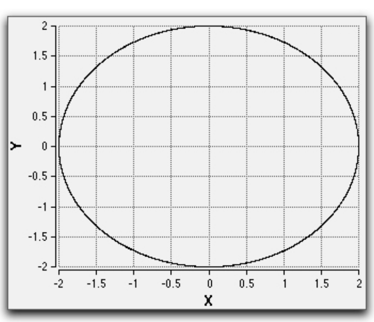
The simplest program to solve this system is:

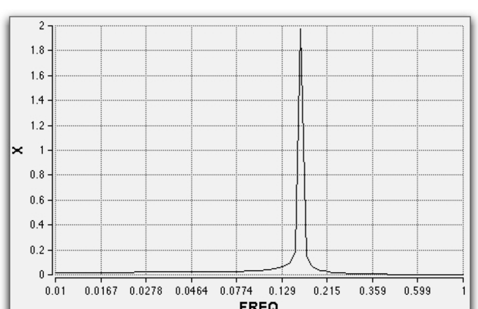
```
METHOD EULER
STARTTIME = 0
STOPTIME = 100
DT = 0.01
d/dt (X) = v
d/dt (v) = -K*X
INIT v = 0
INIT X = 2
K = 1
```



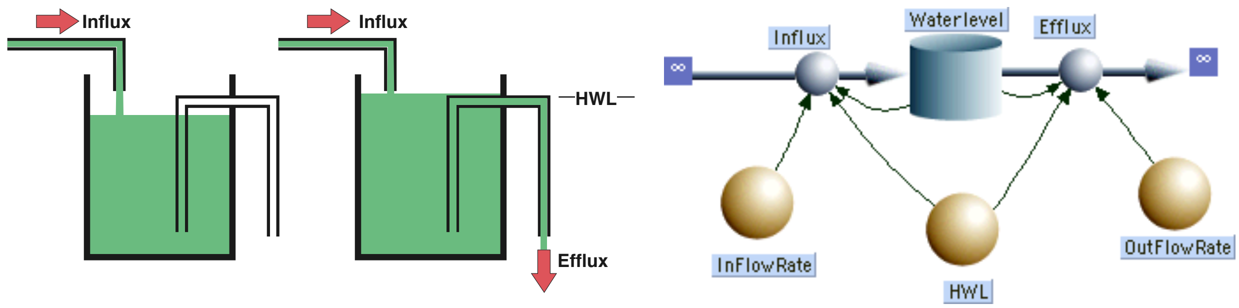


But the oscillation grows! The culprit is the Euler method, so switch to, say, RK4. A better way to plot the oscillation is to use the *phase plane* (x, v), on which the trajectory is a circle. The frequency of the oscillation can be gotten by pressing the Fourier Transform button and changing the x-axis to a log scale.





Exercise 1. Here is a simple oscillating system based on the siphon principle: When the fluid level reaches the High Water Level (HWL) the siphon empties the reservoir (i.e. Efflux >> Influx). Program the siphon oscillator using IF…THEN statements.



Exercise 2. The 'Brusselator'. The following system is a model for an oscillating chemical reaction. Find the approximate value for b for which the system becomes a limit cycle. Use the Rosenbrock stiff solver with DT = 0.002, DTMAX = 1, TOLORANCE = 0.01. Find the period of the oscillation using the Fourier transform button on the Graph window.

$$\frac{dx}{dt} = \left[1-(b+1)x\right] + ax^2 y, \qquad x(0)=1$$
$$\frac{dy}{dt} = bx - ax^2 y, \qquad y(0)=1 \qquad (1)$$

A bistable switch

The first ingredient of an excitable system is a bistable switch. Consider the first order system: $\frac{dx}{dt} = f(x)$, where f(x) has the shape shown in Figure 1. If the system is perturbed in either direction from its stable points past the unstable point, then it quickly switches to the other equilibrium. A light switch works like this. f(x) defines a '*vector field*' on the line showing which way the system will evolve when perturbed away from its stationary points. We will see that, by coupling this system to another 'slow' variable, one can convert the bistable system into an excitable or oscillatory system.

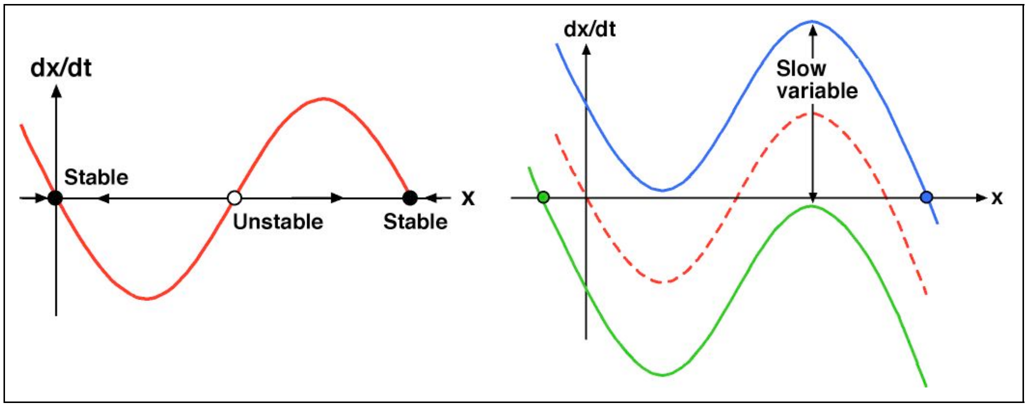
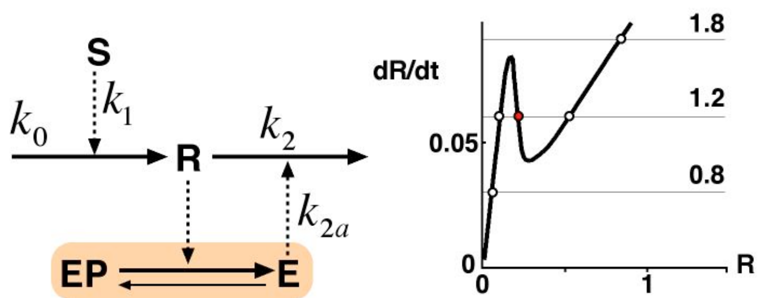


Figure 1. A bistable switch. A commonly used analytic form for $f(x)$ is a cubic polynomial: $f(x) = x(x-1)(x-2)$. This has an unstable root at x = 1, and stable points at x = 0, 2. (Alternative: $f(x) = -x^3/3 + x$, which has an unstable root at x = 0.)

Covalent modification can produce a bistable switch, as shown in Figure 2.



STARTTIME		0
STOPTIME	*	1000
DTMIN		1e-6
DTMAX		1
DTOUT		0
TOLERANCE		0.01
INIT R		5
k0		0
k1	*	4
S		2
k2		0.01
k2p		0.1
INIT E		0.9
ET		1
k3	*	0.05
Km3	*	0.28
k4	*	0.6
Km4	*	0

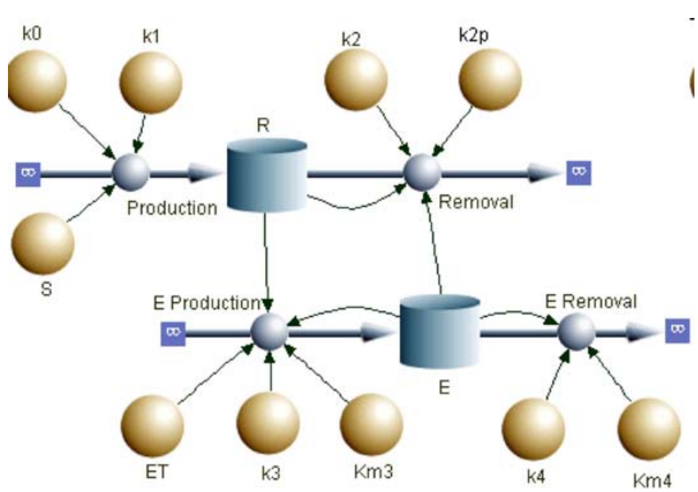
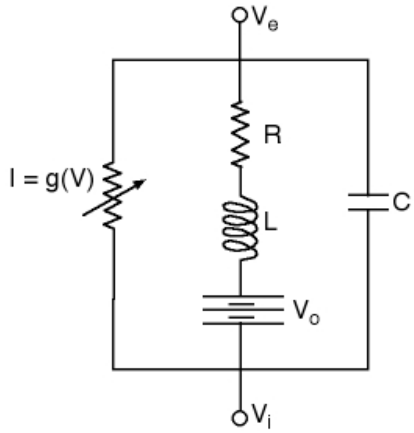


Figure 2. Bistable switch via mutual inhibition circuit.

Exercise 3: Figure 2 is taken from Figure 1f in reference [1]. Using the parameters given here, construct the model and plot the oscillations and the stimulus-response curve.

The Fitzhugh-Nagumo Equations



The best example of an excitable phenomenon is the firing of a nerve: according to the Hodgkin and Huxley equations a sub-threshold depolarization dies away monotonically, but a super-threshold depolarization initiates a spike potential. Fitzhugh and Nagumo devised a simplified version of the H-H equations that describes the essential features of the nerve impulse by only two differential equations.

The ionic current that flows through a nerve membrane is controlled by channels whose openings and closings are controlled by the local electrical field (voltage gated ion channels). For such a conductor, Ohms Law has the form I = g(v), where v is the transmembrane voltage and g(v) is the voltage-dependent conductance. Since Q = C·v , applying d/dt to each side the differential equation for the voltage change is:

$$C\frac{\mathrm{dv}}{\mathrm{dt}} = \frac{dQ}{dt} = I = -\mathrm{g(v)} \qquad (2)$$

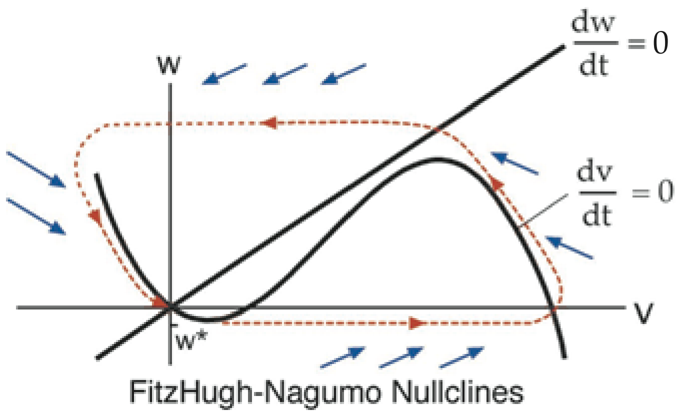
where C is the membrane capacitance and I = dQ/dt is the current. The voltage gate can be either open or shut; that is the conductance is *bistable*, so it has the S-shape shown in Figure 1a.

To turn the bistable conductance equation into an excitable system, Fitzhugh defined a *slow depolarization* variable, w(t), that can move the bistable curve up an down as shown in Figure 1b. This results in the following system:

$$\frac{dv}{dt} = -g(v) - w + I \qquad (1.3)$$

$$\frac{dw}{dt} = \frac{1}{\tau}\left(v - kw - b\right) \qquad (1.4)$$

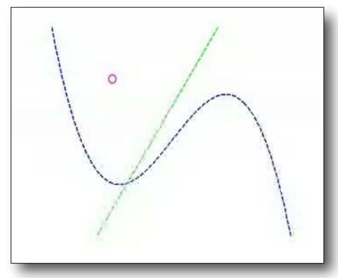
where $\tau > 1$, and $k > 0$.




```
METHOD RK4

STARTTIME = 0
STOPTIME=2000
DT = 0.2

g = v*(v - v0)*(v -1)
v' = - g - w + I
INIT v = 0.2

w' = (v - k*w - b) / tau
INIT w = -0.1

v0   = 0.15
I     = 0
tau  = 40
k     = 0.5
b     = 0

vnullcline = - v*(v - v0)*(v - 1) + I
wnullcline = (v - b) / k
```

Figure 3. Phase plane for equations (2) and (3) showing the Nullclines that lead to excitable behavior.

The *phase portrait* for this system shows how an excitable system works: the single equilibrium at the origin is *locally* stable, but a small perturbation causes the system to make a large excursion before returning to rest. This sort of phase portrait is typical of excitable systems.



Note that by varying a parameter (e.g. *I*) the excitable system can be transformed into a bistable system in two variables. We will also see that, by adjusting the parameters, the system can oscillate in a *limit cycle*.

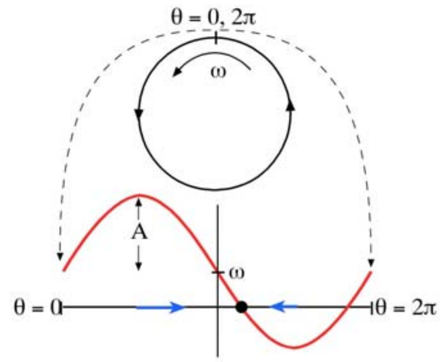
Exercise 4. Use the model equations at the right to make *time* and *phase plane* (w vs. v) plots and then

1. Make sliders for the parameters and find a parameter combination that makes the system oscillate.
2. Make a parameter plot of a critical parameter *I* vs. the amplitude of the oscillation to find the '*bifurcation point*' where the oscillations suddenly appear.
3. Use the initial condition button, Ic, on the graph window to explore the pattern of trajectories.
4. Use the *Fourier Transform* button to estimate the period and frequency of the oscillation.
5. Try the RK2 and Stiff solver methods and compare how many iterations Madonna™ had to execute.

The simplest limit cycles

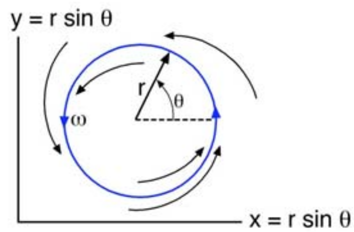
It is sometimes easier to think of periodic phenomena as taking place on a circle: $0 \le \theta \le 2\pi$: $d\theta/dt = \omega(\theta)$. Let $\omega(\theta) = \omega - A \cdot \sin(\theta)$. Sketch the vector field on the circle showing the stability of the equilibrium points and their stability as ω is varied. To do this, 'snip' the circle at $\theta = 0$ and unwrap it so it looks like this ⟶

(Make the length of the vectors proportional to the speed of the 'phase point'.)



A slightly more elaborate version of the circular limit cycle is

$$\frac{dr}{dt} = r(1-r), \quad \frac{d\theta}{dt} = \omega$$



where the radius of the limit cycle, *r*, is governed by the simple logistic equation with amplitude = 1, and the speed around the cycle is ω = constant.

Calcium Oscillations and Cellular Signaling

Here we will learn how to model the oscillatory dynamics of the calcium second messenger system. The reference paper for this problem set is [2]. A reprint is on the course web site.

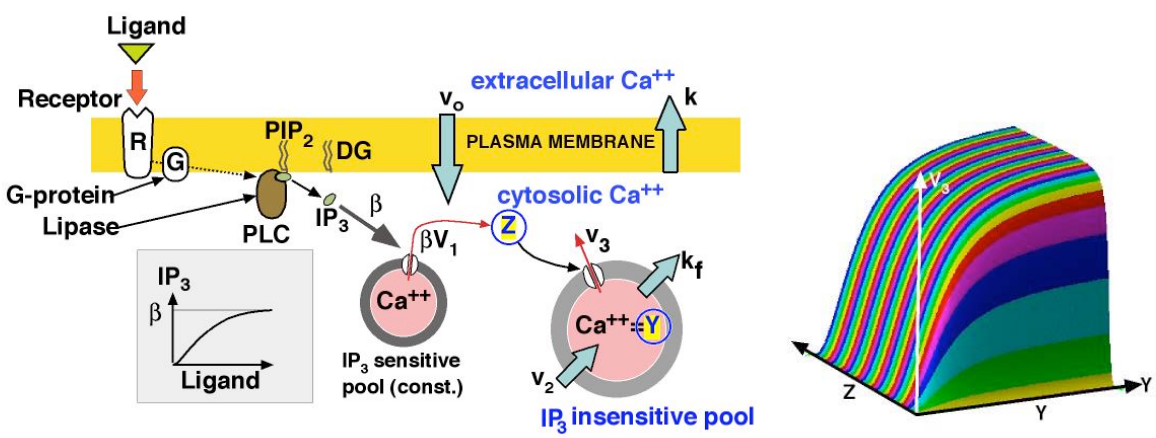


Figure 4. (a) The calcium oscillator. (b) The shape of the reaction velocity functions.

Many types of cells, when stimulated by hormones or neurotransmitters, burst into repetitive spikes of intracellular calcium release. The period of these oscillations ranges from less than 1 second to more than 30 minutes. These oscillations are thought to be an important attribute of intra and intercellular signaling mechanisms. From our viewpoint they are a good example of "limit cycle" kinetics, and will give us an opportunity to learn how to model periodic chemical dynamics.

Consider the calcium transport system, shown in Figure 4. We write conservation equations for the concentration of intracellular calcium, **Z**, and the concentration in the IP‹-insensitive pool (pool 2), **Y**:

$$\underbrace{\frac{dZ}{dt}}_{\substack{\text{rate of change}\\\text{of cytosolic}\\\text{calcium}}} = \underbrace{v_0}_{\substack{\text{into}\\\text{cell}}} + \underbrace{v_1\beta}_{\substack{\text{discharge}\\\text{from pool 1}}} - \underbrace{v_2}_{\substack{\text{transport}\\\text{into pool 2}}} + \underbrace{v_3}_{\substack{\text{transport}\\\text{out of pool 2}}} + \underbrace{k_f Y}_{\substack{\text{leak from}\\\text{pool 2}}} - \underbrace{kZ}_{\substack{\text{transport}\\\text{out of cell}}} \tag{5}$$

$$\underbrace{\frac{dY}{dt}}_{\substack{\text{rate of change}\\\text{of calcium in}\\\text{pool 2}}} = \underbrace{v_2}_{\substack{\text{transport}\\\text{into pool 2}}} - \underbrace{v_3}_{\substack{\text{transport}\\\text{out of pool 2}}} - \underbrace{k_f Y}_{\substack{\text{leak from}\\\text{pool 2}}} \tag{6}$$

The fluxes into and out of the IP3 insensitive pool (2) are the key nonlinearities controlling the behavior of the system. They are Michaelis-Menten type rate laws:

$$v_2 = V_{M2}\frac{Z^n}{K_2^n + Z^n} = 65\frac{Z^2}{1+Z^2} \tag{7}$$

$$\begin{aligned} v_3 &= V_{M3}\frac{Y^m}{K_R^m + Y^m}\cdot\frac{Z^p}{K_A^p + Z^p} \\ &= 500\frac{Y^2}{2^2 + Y^2}\frac{Z^4}{0.9 + Z^4} \end{aligned} \tag{8}$$

Table 1 lists the parameters of the model, their units, and the values that produce oscillatory behavior.

Exercise 5. Make a Madonna Flowchart to simulate the system.

1. Show that the period of the oscillations decreases as β increases.

2. Start with a small value of the composite parameter ($v_o + \beta v_1$) and show that as this quantity increases oscillations begin to appear only after a critical value is reached (this is called a "bifurcation point").

3. Note that the *nullclines* (dZ/dt = 0 = dY/dt) of the calcium regulation system look very such like those of the Fitzhugh-Nagumo equations. Indeed, an examination of the nullclines shows that, with appropriate tuning of parameters, the calcium model can exhibit excitable behavior.

Table 1. **Parameter values**

PARAMETER	VALUE	UNITS
v_o	1	µM/s
k	10	1/s
k_f	1	1/s
v_1	7.3	µM/s
V	65	µM/s
V_{M3}	500	µM/s
K_2	1	µM
K_R	2	µM
K_A	0.9	µM
m	2	1
n	2	1
p	4	1
Y_o	0.1	µM
Z_o	10	µM
β	0.3	

References

1. Tyson, J., Chen, K., and Novak, B. (2003). Sniffers, buzzers, toggles, and blinkers: dynamics of regulatory and signaling pathways in the cell. Curr Opin Cell Biol. *15*, 221-231.

2. Goldbeter, A., Dupont, G., and Berridge, M.J. (1990). Minimal model for signal-induced Ca^{++} oscillations and for their frequency encoding through protein phosphorylation. Proc Natl Acad Sci U S A *87*, 1461-1465.

A good elementary textbook on modeling of dynamical systems in biology is:

- Edelstein-Keshet, L. (1988). *Mathematical Models in Biology*. Ed.) New York: Random House.

Nullcline Analysis

http://www.sosmath.com/diffeq/system/qualitative/qualitative.html

http://www.sosmath.com/diffeq/system/nonlinear/linearization/linearization.html

Qualitative Analysis

Very often it is almost impossible to find explicitly of implicitly the solutions of a system (specially nonlinear ones). The qualitative approach as well as numerical one are important since they allow us to make conclusions regardless whether we know or not the solutions.

Recall what we did for autonomous equations. First we looked for the equilibrium points and then, in conjunction with the existence and uniqueness theorem, we concluded that non-equilibrium solutions are either increasing or decreasing. This is the result of looking at the sign of the derivative. So what happened for autonomous systems? First recall that the components of the velocity vectors are $\frac{dx}{dt}$ and $\frac{dy}{dt}$. These vectors give the direction of the motion along the trajectories. We have the four natural directions (left-down, left-up, right-down, and right-up) and the other four directions (left, right, up, and down). These directions are obtained by looking at the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and whether they are equal to 0. If both are zero, then we have an equilibrium point.

Example. Consider the model describing two species competing for the same prey

$$\begin{cases} \dfrac{dx}{dt} &= x(1-x) - xy \\ \dfrac{dy}{dt} &= 2y(1-y/2) - 3xy \end{cases}$$

Let us only focus on the first quadrant $x \geq 0$ and $y \geq 0$. First, we look for the equilibrium points. We must have

$$\begin{cases} x(1-x) - xy &= 0 \\ 2y(1-y/2) - 3xy &= 0 \end{cases}$$

Algebraic manipulations imply

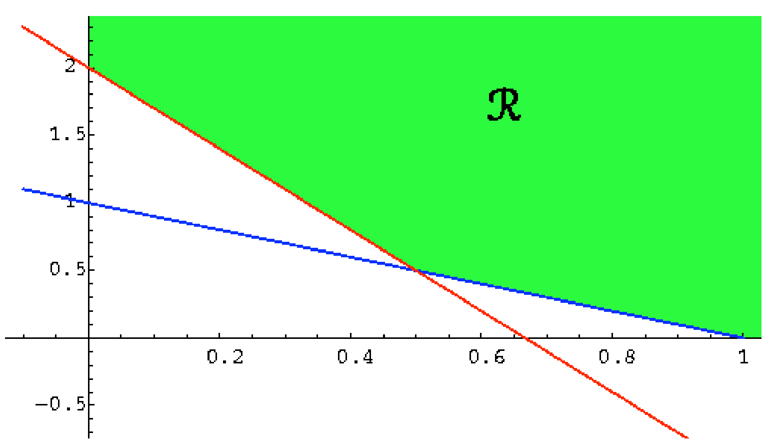
$$x = 0 \ \text{ or } \ 1 - x - y = 0$$

and

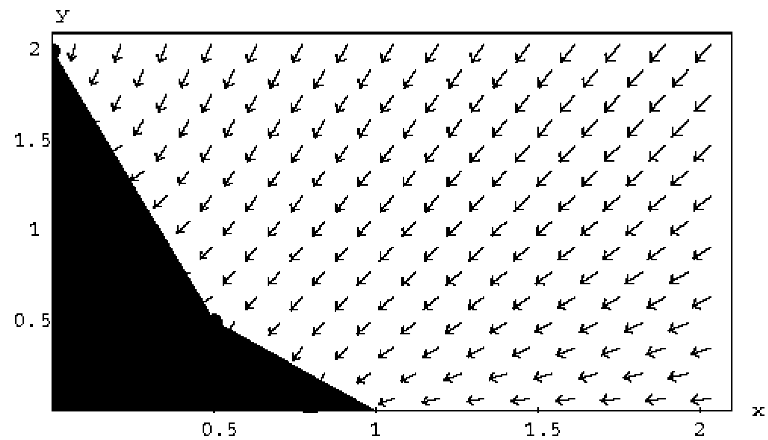
$$y = 0 \ \text{ or } \ 2 - 3x - y = 0$$

The equilibrium points are (0,0), (0,2), (1,0), and $\left(\frac{1}{2}, \frac{1}{2}\right)$.

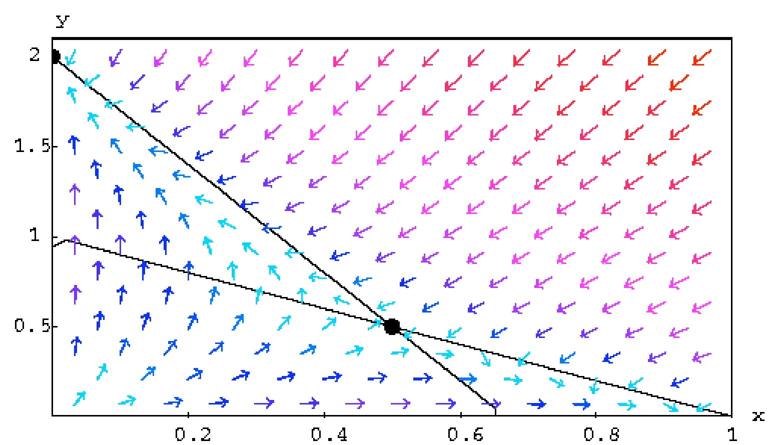
Consider the region **R** delimited by the x-axis, the y-axis, the line 1-*x*-*y*=0, and the line 2-3*x*-*y*=0.



Clearly inside this region neither $\frac{dx}{dt}$ or $\frac{dy}{dt}$ are equal to 0. Therefore, they must have constant sign (they are both negative). Hence the direction of the motion is the same (that is left-down) as long as the trajectory lives inside this region.



In fact, looking at the first-quadrant, we have three more regions to add to the above one. The direction of the motion depends on what region we are in (see the picture below)



The boundaries of these regions are very important in determining the direction of the motion along the trajectories. In fact, it helps to visualize the trajectories as slope-field did for autonomous equations. These boundaries are called **nullclines**.

Nullclines.

Consider the autonomous system

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$

The **x-nullcline** is the set of points where $\frac{dx}{dt} = 0$ and **y-nullcline** is the set of points where $\frac{dy}{dt} = 0$. Clearly the points of intersection between x-nullcline and y-nullcline are exactly the equilibrium points. Note that along the x-nullcline the velocity vectors are vertical while along the y-nullcline the velocity vectors are horizontal. Note that as long as we are traveling along a nullcline without crossing an equilibrium point, then the direction of the velocity vector must be the same. Once we cross an equilibrium point, then we may have a change in the direction (from up to down, or right to left, and vice-versa).

Example. Draw the nullclines for the autonomous system and the velocity vectors along them.

$$\begin{cases} \dfrac{dx}{dt} &= x(1-x) - xy \\[2ex] \dfrac{dy}{dt} &= 2y(1-y/2) - 3xy \end{cases}$$

The x-nullcline are given by

$$\frac{dx}{dt} = x(1-x) - xy = 0$$

which is equivalent to

$$x = 0 \ \text{ or } \ 1 - x - y = 0$$

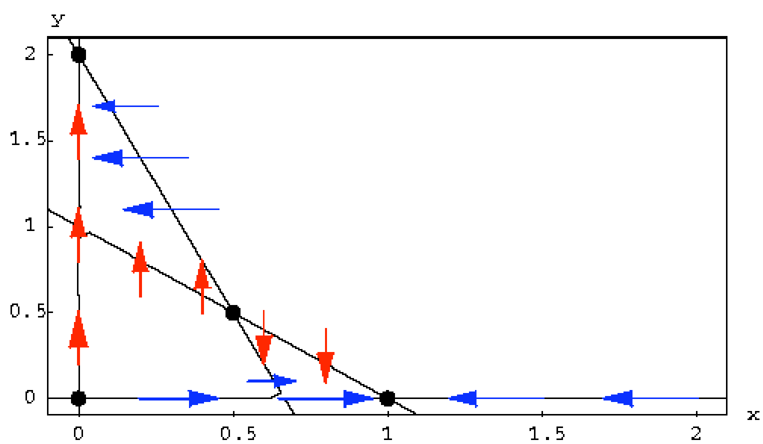
while the y-nullcline are given by

$$\frac{dy}{dt} = 2y(1-y/2) - 3xy = 0$$

which is equivalent to

$$y = 0 \ \text{ or } \ 2 - 3x - y = 0$$

In order to find the direction of the velocity vectors along the nullclines, we pick a point on the nullcline and find the direction of the velocity vector at that point. The velocity vector along the segment of the nullcline delimited by equilibrium points which contains the given point will have the same direction. For example, consider the point (2,0). The velocity vector at this point is (-1,0). Therefore the velocity vector at any point (x,0), with $x > 1$, is horizontal (we are on the y-nullcline) and points to the left. The picture below gives the nullclines and the velocity vectors along them.



In this example, the nullclines are lines. In general we may have any kind of curves.

Example. Draw the nullclines for the autonomous system

$$\begin{cases} \dfrac{dx}{dt} &= x(1-x) - xy \\[2ex] \dfrac{dy}{dt} &= 2y(1-y^2/2) - 3x^2y \end{cases}$$

The x-nullcline are given by

$$\frac{dx}{dt} = x(1-x) - xy = 0$$

which is equivalent to

$$x = 0 \ \text{ or } \ 1 - x - y = 0$$

while the y-nullcline are given by

$$dy$$

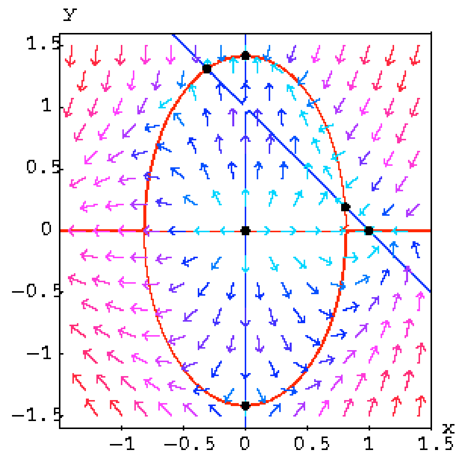
$$\frac{dy}{dt} = 2y(1 - y^2/2) - 3x^2y = 0$$

which is equivalent to

$$y = 0 \quad \text{or} \quad 2 - 3x^2 - y^2 = 0$$

Hence the y-nullcline is the union of a line with the ellipse

$$3x^2 + y^2 = 2$$



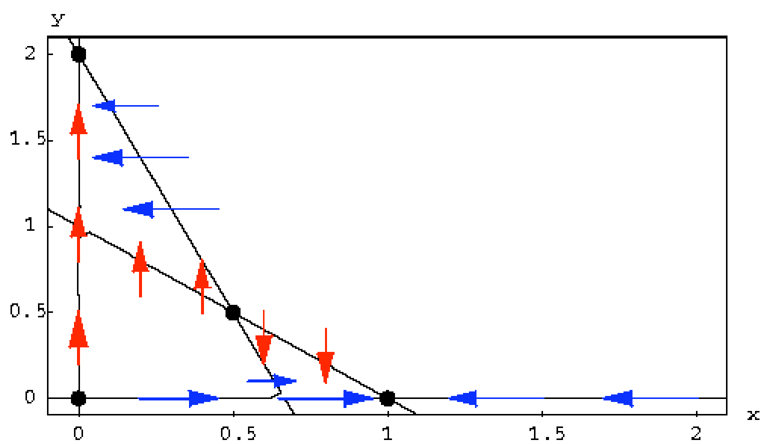
Information from the nullclines

For most of the nonlinear autonomous systems, it is impossible to find explicitly the solutions. We may use numerical techniques to have an idea about the solutions, but qualitative analysis may be able to answer some questions with a low cost and faster than the numerical technique will do. For example, questions related to the long term behavior of solutions. The nullclines plays a central role in the qualitative approach. Let us illustrate this on the following example.

Example. Discuss the behavior of the solutions of the autonomous system

$$\begin{cases} \dfrac{dx}{dt} &= x(1-x) - xy \\ \dfrac{dy}{dt} &= 2y(1-y/2) - 3xy \end{cases}$$

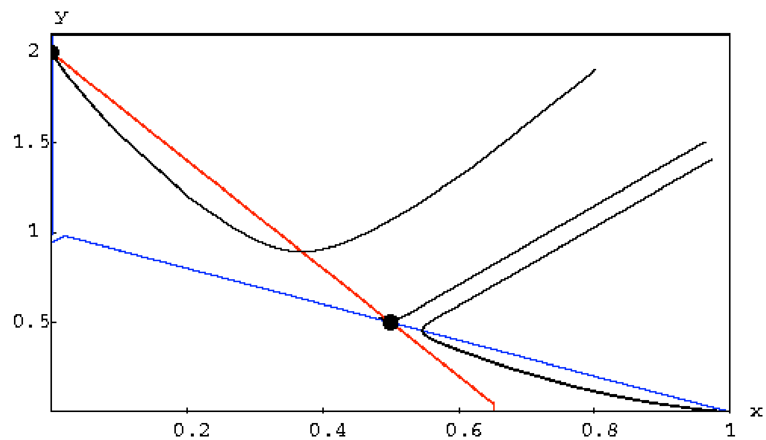
We have already found the nullclines and the direction of the velocity vectors along these nullclines.



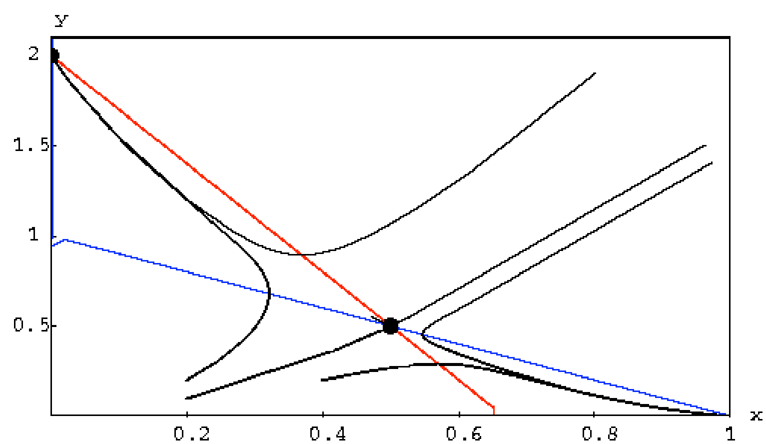
These nullclines give the birth to four regions in which the direction of the motion is constant. Let us discuss the region bordered by the x-axis, the y-axis, the line $1-x-y=0$, and the line $2-3x-y=0$. Then the direction of the motion is left-down. So a moving object starting at a position in this region, will follow a path going left-down. We have three choices

- **First choice:** the trajectory dies at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

- **Second choice:** the starting point is above the trajectory which dies at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Then the trajectory will hit the triangle defined by the points $\left(\frac{1}{2}, \frac{1}{2}\right)$, (0,1), and (0,2). Then it will go up-left and dies at the equilibrium point (0,2).
- **Third choice:** the starting point is below the trajectory which dies at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Then the trajectory will hit the triangle defined by the points $\left(\frac{1}{2}, \frac{1}{2}\right)$, (1,0), and $\left(\frac{2}{3}, 0\right)$. Then it will go down-right and dies at the equilibrium point (1,0).



For the other regions, look at the picture below. We included some solutions for every region.



Remarks. We see from this example that the trajectories which dye at the equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$ are crucial to predicting the behavior of the solutions. These two trajectories are called **separatrix** because they separate the regions into different subregions with a specific behavior. To find them is a very difficult problem. Notice also that the equilibrium points (0,2) and (1,0) behave like sinks. The classification of equilibrium points will be discussed using the approximation by linear systems.

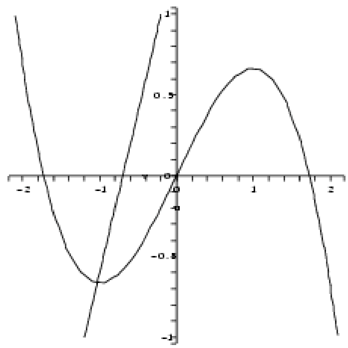
The FitzHugh-Nagumo model which we will use for the analysis is:

$$\dot{u} = \frac{(v + u - u^3/3)}{\epsilon}$$
$$\dot{v} = \epsilon(u + \beta - \gamma v)$$

The following formulas

$$v = \frac{1}{3}u^3 - u$$
$$v = \frac{1}{\gamma}u + \frac{\beta}{\gamma}$$

are the u and v nullcline respectively. *(See graph of the nullcline below.)*



Graph of the nullcline with parameter $(\epsilon = 0.2, \beta = 0.7, \gamma = 0.5)$

The nullclines $v = \frac{1}{3}u^3 - u$ and $v = \frac{1}{\gamma}u + \frac{\beta}{\gamma}$ represent the set of points when the change with repect to time of u and v is null. Hence the solution will cross the 'cubic' nullcline *(i.e. u nullcline)* when travelling vertically while the solution will cross the 'line' nullcline *(i.e. v nullcline)* when travelling horizontally. Suppose that we fix the parameters of the FitzHugh-Nagumo model to $\epsilon = 0.2, \beta = 0.7, \gamma = 0.5$, then suppose that you let a solution start at $x_0 = (1, -0.5)$. Then we notice that $\frac{du}{dt} > 0$ and that $\frac{dv}{dt} > 0$, so the solution will travel upwards and to the right. It will only cross the 'cubic' nullcline vertically. Therefore once the solution will be close to the nullcline it will travel vertically and cross the nullcline. Once the solution has crossed the nullcline it will move upwards and to the left since $\frac{du}{dt} < 0$ and that $\frac{dv}{dt} > 0$. It will then cross the 'line' nullcline horizontally and the solution will move downwards and will remain moving towards the left since $\frac{du}{dt} < 0$ and $\frac{dv}{dt} < 0$, until the solution reaches the 'cubic' nullcline which it will cross vertically. Once the solution has crossed the last nullcline it will continue to move downwards but will now progressively start moving to the right since $\frac{du}{dt} > 0$ and $\frac{dv}{dt} < 0$. This is all that can be said about this specific case using the analysis of the nullcline. Below is an example of the solution. As we can see it intially behaves like the above description.